First Order Sliding Mode Controller Design for Inverted Pendulum System

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Abstract: This paper presents first order sliding mode controller and input-output linearization technique for the control of inverted pendulum system. Sliding mode controller is considered as an efficient tool for studying different systems due to its accuracy and robustness to disturbances. The stability analysis is performed by Lyapunov function for the controllers. The performance of proposed controllers has been demonstrated by comparing the results of cart position and angular position of inverted pendulum with each other.

Keywords: Inverted pendulum, input-output linearization, Lyapunov function, sliding mode controller, underactuated systems.

Introduction

Underactuated systems find ready applications in robotics and automobile field. These systems are having less number of actuators than their degrees of freedom [1]. There is a difficulty in controlling these systems because the techniques which are developed for the fully actuated systems cannot be directly used. Also these systems are not feedback linearizable [2]. The underactuated systems present challenging control problems. One of the common methods in controlling underactuated systems is the use of sliding mode controller based on the Lyapunov design.

The sliding mode controller (SMC) has been considered as best approach due to its robustness to the disturbances and its high accuracy. It consists of two steps: the first is to choose a surface that forces the trajectories to remain along the sliding surface and the second is to choose a state feedback which is capable of forcing the state variables to reach the surface in a finite time. The only drawback of SMC is the chattering effect which is caused by high frequency oscillations of the proposed controller [3]. These high frequency components may degrade the system performance and can also lead to instability of the system. There are three methods to control the chattering effect.

The first one is to use the saturation control instead of the discontinuous control. It ensures the convergence to a boundary layer of the sliding surface. Although it ensures the convergence, the accuracy and the robustness of the sliding mode are partially lost. The second one is the use of observer based approach [4]. It can approximate the robust control problem to the robust estimation. This can lead to deterioration of the robustness with respect to the uncertain disturbances. The third one is the use of higher order sliding mode control [5,6] however the second order is frequently used due to its simple structure and gives good robustness to the disturbances. The stability and convergence of the second order sliding mode controller is a challenging task to face the difficulties with disturbances. However Lyapunov function provides the stability and finite time convergence of the system.

Inverted pendulum is considered as a typical nonlinear underactuated system because of its instability at its upright position. For this system the control input is the force u which is used to move the cart horizontally and the output is the angular deviation of the pendulum from its upright position θ. Despite from its simple structure the system needs sophisticated methods to control it. Also it is proven that the system is not feedback linearizable and has no constant relative degree. The main difficulty exists in moving the pendulum from its stable downward position to the unstable upright position by keeping the cart stable. The main objective is to develop a robust controller based on the first order SMC applied to inverted pendulum. Stability is carried out by using the Lyapunov function for the proposed controller [7].

Input-output linearization technique is also used to control the inverted pendulum. With a high gain the controller used in the input-output linearization, the system becomes singly perturbed with respect to zero dynamics. This implies the dynamics under this technique has a fast transient and therefore the zero dynamics can be treated as independent system. This paper is organized as follows. Section 2 explains the mathematical modeling of the inverted pendulum system and the first order sliding mode controller. Section3 deals with the input-output linearization technique. Section 4 describes the simulation results of the proposed controllers.

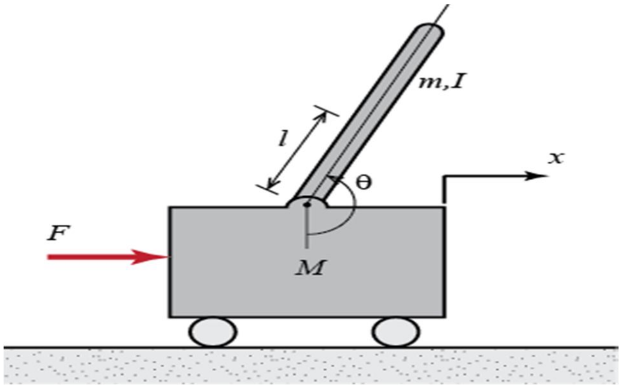

Figure1. Inverted Pendulum [8]

Mathematical model and control approach of an inverted pendulum

Mathematical model

The inverted pendulum is a single input multi output system having one input and two outputs. Gears in the motor couple the rotation of the motor into linear motion of the cart. The behavior of an inverted pendulum is described by the following equations:

$$(M+m)\ddot{x} + (B_{eq} + K_2)\dot{x} - ml\ddot{\theta}\cos\theta + ml\dot{\theta}^2\sin\theta = K_1 u$$

$$(J + ml^2)\ddot{\theta} + B_p\dot{\theta} - m\ddot{x}l\cos\theta - mgl\sin\theta = 0 \qquad (1)$$

Where, l = length of the pendulum,
m = mass of the pendulum,
M = mass of the cart,
$K_1 u$ = horizontal force,
θ = deviation of the pendulum from its upright position,
x = cart position.

To express the differential equation into a state space, the following state variables for the system are defined as:

$$x_1 = x,\ x_2 = \dot{x},\ x_3 = \theta,\ x_4 = \dot{\theta}$$

Let,

$$g_2(x) = \frac{a_1 k_1}{m^2 l^2 \sin^2 x_3 + a_4};\ g_4(x) = \frac{k_1 ml\cos x_3}{m^2 l^2 \sin^2 x_3 + a_4}$$

$$a_1 = (J + ml^2),$$

$$a_2 = (Jml + m^2 l^3),$$

$$a_3 = m^2 l^2 g,$$

$$a_4 = Mml^2 + J(M+m),$$

$$a_5 = (M+m)mgl$$

From eq (1) by subjecting x and θ while substituting their values in another equation we can get eq (2) as

$$f_2(x) = \frac{-a_1 B_{eq} x_2 - mlB_p x_4\cos x_3 - a_2 x_4^2 \sin x_3 + a_3\sin x_3\cos x_3 - a_1 K_2 x_2}{m^2 l^2 \sin^2 x_3 + a_4} + g_2(x)u$$

$$f_4(x) = \frac{-(M+m)B_p x_4 - m^2 l^2 x_4^2 \sin x_3\cos x_3 - mlB_{eq} x_2\cos x_3 + a_3\sin x_3 - k_2 ml x_2\cos x_3}{m^2 l^2 \sin^2 x_3 + a_4} + g_4(x)u \qquad (2)$$

Finally, the state space representation of the inverted pendulum with actuator is represented as

$\dot{X} = F(x) + G(x)u,\ y = H(x)$

(3)

where, $H(x)=\begin{bmatrix} x_1 \\ x_3 \end{bmatrix}$, $F(x)=\begin{bmatrix} x_2 \\ f_2(x) \\ x_4 \\ f_4(x) \end{bmatrix}$, $G(x)=\begin{bmatrix} 0 \\ g_2(x) \\ 0 \\ g_4(x) \end{bmatrix}$ (4)

Voytsekhovsky and Hirschorn [9] introduced a method that can make the original system appropriate to an input-output linearizable system with new coordinates. This is based upon the coordinate transformation by mapping $T : x \rightarrow z$ whereas z is defined by $z_i = L_f^{i-1} h(x)$ i∈ {1, 2, 3, and 4}; T is defined as local diffeomorphism. The dynamic system of the model can be approximated as follows

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= z_3 \\ \dot{z}_3 &= z_4 \\ \dot{z}_4 &= L_f^4 (T^{-1}(z)) + L_q L_f^3 h(T^{-1}(z))u \end{aligned} \quad (5)$$

where, $L_f\ h(x)$ is the Lie derivative of h(x), by definition of the output system in eq (3)

$$z(x) = H(x) = x_1 - l \log(\tan x_3 + \sec_3) \quad (6)$$

With the output y=h(x) we proceed to find diffeomorphism

$$z = \begin{bmatrix} y \\ \dot{y} \\ \ddot{y} \\ \dddot{y} \end{bmatrix} \quad (7)$$

By differentiating eq (6) we can get z(x). As a result, the approximate linearized system is given below

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= z_3 \\ \dot{z}_3 &= z_4 \\ \dot{z}_4 &= f_e(x) + g_e(x)u \end{aligned} \quad (8)$$

First order sliding mode controller

By defining the sliding surface as $s(e) = \left(\frac{d}{dt} + \lambda\right)^3 e$

The derivative of the sliding surface along the system trajectory becomes

$$\dot{s} = \dddot{z} + 3\lambda\ddot{z} + 3\lambda^2\dot{z} + \lambda^3 z = = f_e(x) + g_e(x)w + 3\lambda z_4 + 3\lambda^2 z_3 + \lambda^3 z_2 \quad (9)$$

Lyapunov function based on the sliding surface is chosen as V=0.5s^2 (10)

The definition of the control action is specified in a way that the derivative of the Lyapunov function should be negative definite. So, for $\dot{V}(s) = s\dot{s}$ to be negative definite s and $\dot{s}$ should be of opposite sign. This is a fundamental condition for the system to reach the sliding surface. Various laws that meet this reaching condition and these are called as reaching laws.

In this paper, we consider the power reaching law having the form

$$\dot{S} = (k|S|^{\alpha}\, sign(s))g_e(x) \tag{11}$$

The sliding mode control action involves in evaluating the $\dot{S}$ and equating it to the reaching law. To deal with the chattering phenomena the continuous functions such as saturation and relay functions are approximated to the discontinuous functions such as sign functions.

The sliding mode control is defined as

$$u = u_{eq} + u_{sw} \tag{12}$$

where, $$u_{eq} = -\frac{f_e(x)+3\lambda z_4+3\lambda^2 z_3+\lambda^3 z_2}{g_e(x)} \tag{13}$$

$u_{sw} = K|S|^{\alpha}\left(\frac{s}{|s|+\varepsilon}\right)$. Because of the Lyapunov function the convergence and stability of the system is guaranteed.

Input-Output Linearization

The aim of the input-output linearization is to obtain a state feedback control that linearizes the map between system output and virtual control input through state transformation. If the relative degree is less than the order of the system, then the nonlinear system is only partially feedback linearized [11] and therefore consists of feedback linearized system controlled by a virtual control input. Then the internal dynamics of a system is defined as:

$$\left.\begin{aligned}
&\dot{z}_1 = \dot{y} = L_f^1 h(x) = z_2\\
&\dot{z}_2 = \ddot{y} = L_f^2 h(x) = z_3\\
&.\\
&.\\
&.\\
&\dot{z}_{n-r} = y^{r-1} = L_f^{r-1} h(x) = z_r\\
&\dot{z}_r = a(z,n)+b(b,n)u = v
\end{aligned}\right\} \tag{15}$$

Thereafter, the virtual control input only affects the feedback linearizable system. Hence the internal dynamics is uncontrolled by the virtual control. To design the input-output linearization control for the output $y=x_3$ we have to differentiate repeatedly until u appears as shown below:

$$\begin{aligned}
&\dot{y} = \dot{x}_3 = x_4\\
&\ddot{y} = \ddot{x}_3 = f_4(x)+g_4(x)u = v_1
\end{aligned} \tag{16}$$

where, v_1 is defined as $v_1 = -k_3x_3 - k_4x_4$. k_3 and k_4 are designed using the pole placement technique. From the above equation we can get u by substituting the values of f_4 and g_4.

Results and Discussions

SMC using approximate linearization and input-output linearization techniques have been designed for the stabilization of inverted pendulum system. The proposed controllers are designed using MATLAB ODE45. In the first order sliding mode controller main disadvantage is chattering in the control input. But it can be further reduced by using second and higher order SMC. In this paper chattering phenomenon is reduced by approximating continuous functions to discontinuous sign functions. Parameters of the inverted pendulum are taken as M = 0.94 kg, m = 0.23 kg, l = 0.3302m, g = 9.8 N/kg. The initial conditions of the cart pendulum are $\left(y_0, \dot{y}_0\right) = (0,0), \left(\theta_0, \dot{\theta}_0\right) = (0.2,0)$ and the desired position for the cart is set as 0.2.

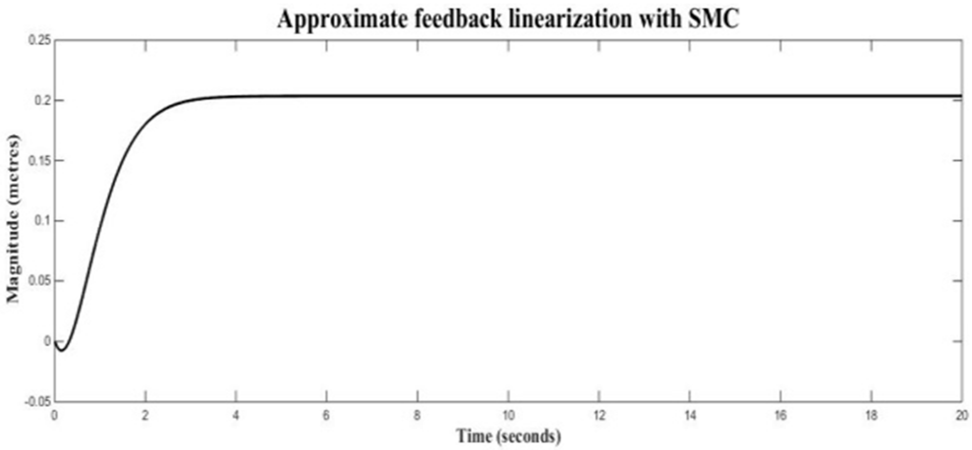

Figure2. Position of the cart for the uncertain system

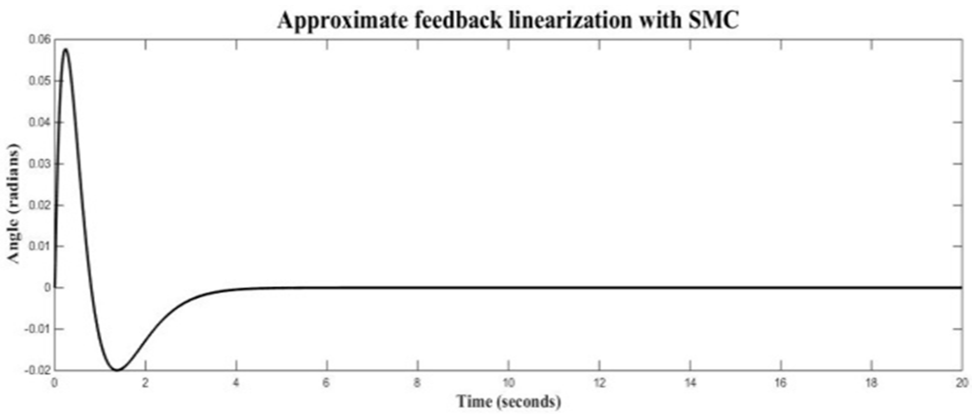

Figure3. Position of the pendulum for uncertain systems

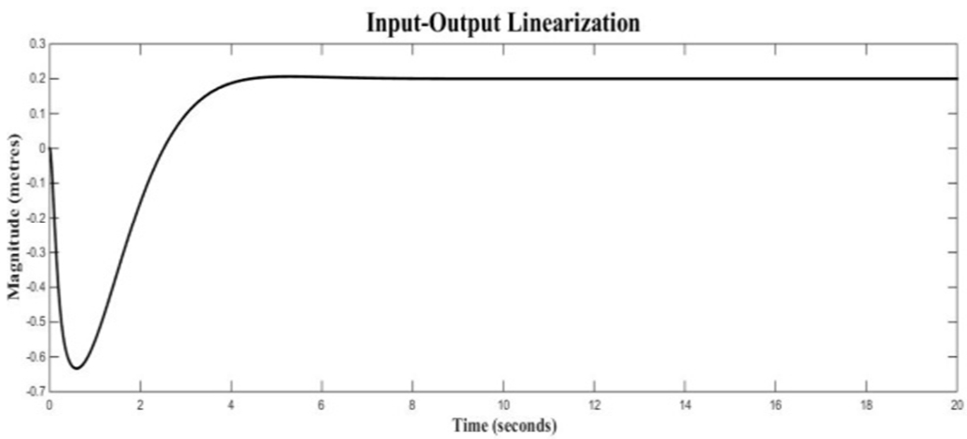

Figure4. Position of the cart for uncertain system

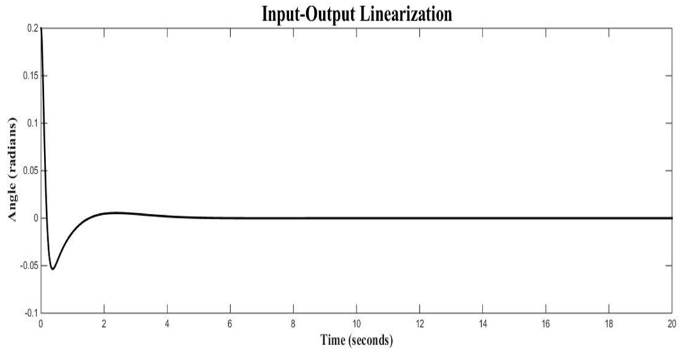

Figure5. Angular position of the pendulum

Simulation results of the proposed controllers are shown by above figures. Results are done by using λ=2.5, k=15, α=0.2 ε =0.1. For input-output linearization the values of k_3 and k_4 are taken as 100.25 and 20. These values are obtained through pole placement technique. The proposed controllers are compared in tabular form. From the figures 2 and 3 it is inferred that the cart has a rise time of 2sec and settling time of 3sec. The pendulum settles in 3.5sec without having any overshoot. Similarly from the figures 4 and 5 it is observed that the settling time of the cart is 4 sec and rise time is 3 sec without having any steady state error. But in comparison SMC gave better performance to input-output linearization technique.

Table 1. Comparison of Results

S. No	Techniques for the inverted pendulum		
	Parameters	SMC	Input-output linearization
1.	Cart referrence	0.2	0.2
2.	Settling time of the cart	3 sec	4 sec
3.	Settling time of the pendulum	3.5 sec	4 sec
4.	Rise time of the cart	2 sec	3 sec
5.	Overshoot	Nil	Nil
6.	Steady state error	Nil	Nil

Conclusions

In this paper first order SMC and input-output linearization technique have been successfully applied for the control of inverted pendulum system. From the results shown through figures it is clearly known that both the controllers met the desired goals with satisfactory performance. But sliding mode controller (SMC) has been found to be more robust to disturbances. The nonlinear controller designed on the feedback linearization has shown the better performance compared to linearization technique.

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